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Cylindrically symmetric cosmological solutions of the Lyttleton–Bondi universe

J R Rao and H S Panda

Department of Mathematics, Indian Institute of Technology, Kharagpur 721302. India

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Abstract. The exterior field of the Einstein–Rosen cylindrically symmetric metric in the Lyttleton–Bondi universe is considered and exact solutions are obtained in some specific situations. The solutions exhibit singularities on the z axis as well as at spatial infinity. The geodesic path of a test particle exhibits a spiral structure around the axis of symmetry.

1. Introduction

Lyttleton and Bondi (1959) have developed a cosmological model assuming that there is a continuous creation of matter due to a net imbalance of charge. This imbalance of charge may arise from the difference in magnitude of the charge of the proton and that of the electron, or from the difference in number of protons as compared to the number of electrons. The imbalance is of such order that it does not affect appreciably the conductivity of a material. Charge and matter once created are conserved.

Further, in a uniform distribution of unionized hydrogen atoms, a hydrogen atom placed at a distance from the centre will experience a net force radially outwards, causing a diminution in the space–time density of matter. In order to keep this density invariable, the idea of creation of matter through the creation of charge is postulated. Since charge is not conserved in the strict sense, a modification of the Maxwell equations is done on the lines formulated by Proca. Lyttleton and Bondi (1959) have investigated the nature of the field in a Newtonian framework with zero electromagnetic field and nonzero potentials. The same assumptions have also been utilized by them to study the De Sitter metric. They have also discussed the probability of light nucleons being expelled with an energy comparable to that of cosmic rays.

Burman has studied several aspects of the Lyttleton–Bondi field. One of his investigations (Burman 1971) pertains to the static spherically symmetric exterior solution. By use of a method suggested by Eddington (1924), he succeeded in showing that the field near a central body has no detectable departure from the predictions of the Schwarzschild metric. In the present work we have formulated the analogous exterior problem in the case of the cylindrically symmetric Einstein–Rosen metric and obtained exact solutions, one of which is static and the other time-dependent. It has been shown that singularities occur on the axis of symmetry and at spatial infinity. The geodesic motion has been determined in the two cases and the possibility of spiral motion around the axis of symmetry is exhibited. It is interesting to note that this phenomenon may possibly explain the spiral structure of galactic formations.

2. Covariant formulation of the field

Let A_μ and J_μ denote four-potential and current density four-vector respectively. $F_{\mu\nu}$ denotes the antisymmetric electromagnetic field tensor and q the rate of creation of charge per unit proper volume. To incorporate the idea of creation, the Maxwell field equations are modified as

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} \tag{1}$$

$$F^{\mu\nu}_{;\nu} = J^\mu - \lambda A^\mu \tag{2}$$

$$J^\mu_{;\mu} = q \tag{3}$$

where λ is a constant and a semicolon denotes covariant differentiation.

The energy–momentum tensor of the field is

$$T_\mu{}^\nu = (F_{\mu\alpha}F^{\alpha\nu} + \tfrac{1}{4}g_\mu{}^\nu F_{\alpha\beta}F^{\alpha\beta}) + \lambda(A_\mu A^\nu - \tfrac{1}{2}g_\mu{}^\nu A_\alpha A^\alpha). \tag{4}$$

2.1. Solutions with zero electromagnetic field

When $F_{\mu\nu} = 0$, relation (2) becomes

$$J_\mu = \lambda A_\mu \tag{5}$$

and the energy–momentum tensor of the field becomes

$$T_\mu{}^\nu = \lambda(A_\mu A^\nu - \tfrac{1}{2}g_\mu{}^\nu A_\alpha A^\alpha). \tag{6}$$

Throughout this paper the fundamental velocity c is taken to be unity and the rationalized relativistic system of units is used.

3. Space–time outside a cylindrically symmetric object

In this section we deal with the space–time outside a cylindrically symmetric object. We assume that the radial outflow of charge created will not affect the dynamical characteristic of the metric; the mechanical effect of such radial outflow on the energy–momentum tensor is also assumed to be nil, and as such we can use Einstein's field equations

$$G_\mu{}^\nu \equiv R_\mu{}^\nu - \tfrac{1}{2}g_\mu{}^\nu R = -\kappa T_\mu{}^\nu \tag{7}$$

where $k(= 8\pi G/c^4)$ is a constant.

We consider the cylindrically symmetric static line element

$$ds^2 = -e^{2\alpha-2\beta}\,d\rho^2 - \rho^2 e^{-2\beta}\,d\phi^2 - e^{2\beta}\,dz^2 + e^{2\alpha-2\beta}\,dt^2 \tag{8}$$

where α and β are functions of ρ only and (ρ, ϕ, z, t) correspond to (x^1, x^2, x^3, x^4) coordinates respectively. Here

$$\begin{aligned} g_{11}, g_{22}, g_{33}, g_{44} &= -e^{2\alpha-2\beta}, -\rho^2 e^{-2\beta}, -e^{2\beta}, e^{2\alpha-2\beta} \\ g^{11}, g^{22}, g^{33}, g^{44} &= -e^{2\beta-2\alpha}, -\rho^{-2} e^{2\beta}, -e^{-2\beta}, e^{2\beta-2\alpha}. \end{aligned} \tag{9}$$

For the metric (8) the mixed components of the Einstein tensor are given by

$$\begin{aligned} G_1^1 &= e^{-2\alpha+2\beta}(\alpha_1/\rho - \beta_1^2) \\ G_2^2 &= e^{-2\alpha+2\beta}(\alpha_{11} + \beta_1^2) \\ G_3^3 &= -e^{-2\alpha+2\beta}[2(\beta_{11} + \beta_1/\rho) - (\alpha_{11} + \beta_1^2)] \\ G_4^4 &= -G_1^1 = e^{-2\alpha+2\beta}(\beta_1^2 - \alpha_1/\rho). \end{aligned} \tag{10}$$

The suffixes 1, 11 after an unknown function denote the first- and second-order partial derivatives of the function with respect to ρ.

In a static cylindrically symmetric situation, A_μ must have the form given by

$$A_\mu = (a, 0, 0, \varphi). \tag{11}$$

But since $F_{\mu\nu} = 0$, φ must be a constant. Using the values of $g^{\mu\nu}$ given in (9) we obtain

$$A^\mu = (-a\, e^{2\beta-2\alpha}, 0, 0, \varphi\, e^{2\beta-2\alpha}) \tag{12}$$

so that

$$A_\mu A^\mu = e^{2\beta-2\alpha}(\varphi^2 - a^2). \tag{13}$$

Equations (6), (7), (10)–(13) give rise to the following field equations:

$$\alpha_1/\rho - \beta_1^2 = \tfrac{1}{2}\kappa\lambda(\varphi^2 + a^2) \tag{14}$$

$$\alpha_{11} + \beta_1^2 = \tfrac{1}{2}\kappa\lambda(\varphi^2 - a^2) \tag{15}$$

$$-2(\beta_{11} + \beta_1/\rho) + (\alpha_{11} + \beta_1^2) = \tfrac{1}{2}\kappa\lambda(\varphi^2 - a^2) \tag{16}$$

$$\lambda a \varphi = 0; \tag{17}$$

equation (17) gives

$$\varphi = 0. \tag{18}$$

Subtracting (16) from (15) leads to

$$\beta_{11} + \beta_1/\rho = 0 \tag{19}$$

which gives

$$\beta = m \ln \rho + n \tag{20}$$

where m and n are constants of integration. Adding (14) and (15) and using (18), we have

$$\alpha_{11} + \alpha_1/\rho = 0, \tag{21}$$

3.1. Non-static case

In this case we assume α, β, a and φ to be functions of ρ and t only. Here also we make such assumptions as before, for which we can use the Einstein field equations (7).

The values of A_μ, A^μ and $A_\mu A^\mu$ will be given by the same expressions as in (11), (12) and (13); but considering $F_{14} = 0$ we get an extra relation of the form

$$\frac{\partial a}{\partial t} = \frac{\partial \varphi}{\partial \rho} \tag{25}$$

in a and φ.

The mixed components of the Einstein tensor are given by

$$G_1^1 = e^{2\beta - 2\alpha}(\alpha_1/\rho - \beta_1^2 - \beta_4^2) \tag{26}$$

$$G_2^2 = e^{2\beta - 2\alpha}(\alpha_{11} - \alpha_{44} + \beta_1^2 - \beta_4^2) \tag{27}$$

$$G_3^3 = -e^{2\beta - 2\alpha}[2(\beta_{11} - \beta_{44} + \beta_1/\rho) + (\alpha_{44} - \alpha_{11} + \beta_4^2 - \beta_1^2)] \tag{28}$$

$$G_4^4 = -G_1^1 = e^{2\beta - 2\alpha}(\beta_1^2 + \beta_4^2 - \alpha_1/\rho) \tag{29}$$

$$G_4^1 = -G_1^4 = -e^{2\beta - 2\alpha}(2\beta_1\beta_4 - \alpha_4/\rho) \tag{30}$$

where the suffixes 4, 44 after an unknown function denote the first- and second-order partial derivatives of the function with respect to t.

Equations (7), (11)–(13) and (26)–(30) yield the following field equations:

$$\alpha_1/\rho - \beta_1^2 - \beta_4^2 = \tfrac{1}{2}\kappa\lambda(\varphi^2 + a^2) \tag{31}$$

$$\alpha_{11} - \alpha_{44} + \beta_1^2 - \beta_4^2 = \tfrac{1}{2}\kappa\lambda(\varphi^2 - a^2) \tag{32}$$

$$-2(\beta_{11} - \beta_{44} + \beta_1/\rho) + (\alpha_{11} - \alpha_{44} + \beta_1^2 - \beta_4^2) = \tfrac{1}{2}\kappa\lambda(\varphi^2 - a^2) \tag{33}$$

$$\alpha_4/\rho - 2\beta_1\beta_4 = \kappa\lambda a\varphi. \tag{34}$$

Subtracting (32) from (33), we get

$$\beta_{11} - \beta_{44} + \beta_1/\rho = 0 \tag{35}$$

which gives

$$\beta = (c_4 t + c_5)\ln\rho \tag{36}$$

where c_4 and c_5 are constants. As (35) in conjunction with (32) yields (33), we ignore (33) and treat (31), (32), (35) and (34) as the field equations.

Since the nonlinear character of the equations involved makes it extremely difficult to find exact solutions, we shall consider only the following special cases:

(i)	$a = 0, \varphi = \text{constant}$	(iv)	$a = \text{constant}, \varphi = 0$
(ii)	$a = 0, \varphi = \varphi(t)$	(v)	$a = a(\rho), \varphi = 0$
(iii)	$a = 0, \varphi = \varphi(\rho, t)$	(vi)	$a = a(\rho, t), \varphi = 0.$

Case (i). $a = 0$, $\varphi = \text{constant}$

Under the above consideration the field equations (31), (32), (35) and (34) turn out to be

$$\alpha_1/\rho - \beta_1^2 - \beta_4^2 = \tfrac{1}{2}\kappa\lambda\varphi^2 \tag{37}$$

$$\alpha_{11} - \alpha_{44} + \beta_1^2 - \beta_4^2 = \tfrac{1}{2}\kappa\lambda\varphi^2 \tag{38}$$

$$\beta_{11} - \beta_{44} + \beta_1/\rho = 0 \tag{39}$$

$$\alpha_4/\rho = 2\beta_1\beta_4. \tag{40}$$

Using equations (36) and (37), we obtain

$$\alpha = \tfrac{1}{4}\kappa\lambda\varphi^2\rho^2 + (c_4 t + c_5)^2 \ln\rho + \tfrac{1}{2}c_4^2\,\rho^2[(\ln\rho)^2 - \ln\rho + \tfrac{1}{2}] + c_8 \tag{41}$$

c_8 being constant. The values of α and β given by (41) and (36) also satisfy (39) and (40).

Case (ii). $a = 0$, $\varphi = \varphi(t)$
The field equations in this case will be given by the same set of equations as (37)–(40). Equation (39) gives, as before,

$$\beta = (c_4 t + c_5)\ln\rho$$

and proceeding exactly as in case (i), we get

$$\alpha = \tfrac{1}{4}\kappa\lambda\varphi^2\rho^2 + (c_4 t + c_5)^2 \ln\rho + \tfrac{1}{2}c_4^2\rho^2[(\ln\rho)^2 - \ln\rho + \tfrac{1}{2}] + c_8.$$

Substituting the above values of α and β in (38) yields

$$\varphi\varphi_{44} + \varphi_4^2 = 0$$

whence we have

$$\varphi^2 = At + B$$

where A and B are constants. Substituting these values of α and β in equation (40) leads to

$$\varphi\varphi_4 = 0$$

which gives

$$\varphi = \text{constant}.$$

Combining these above two values of φ produces the same situation as in case (i), namely $a = 0$ and $\varphi = $ constant.

Case (iii). $a = 0$, $\varphi = \varphi(\rho, t)$
Proceeding exactly as above, we obtain the same values for α and β as in case (ii): φ turns out to be a constant in conformity with the relation (25).

Case (iv). $a = $ constant, $\varphi = 0$
The field equations in this case will be given by

$$\alpha_1/\rho - \beta_1^2 - \beta_4^2 = \tfrac{1}{2}\kappa\lambda a^2 \tag{42}$$

$$\alpha_{11} - \alpha_{44} + \beta_1^2 - \beta_4^2 = -\tfrac{1}{2}\kappa\lambda a^2 \tag{43}$$

$$\beta_{11} - \beta_{44} + \beta_1/\rho = 0 \tag{44}$$

$$\alpha_4/\rho - 2\beta_1\beta_4 = 0. \tag{45}$$

From (44) we have

$$\beta = (c_4 t + c_5)\ln\rho. \tag{46}$$

Equations (42) and (46) give

$$\alpha = \tfrac{1}{4}\kappa\lambda a^2\rho^2 + (c_4 t + c_5)^2 \ln\rho + \tfrac{1}{2}c_4^2\rho^2[(\ln\rho)^2 - \ln\rho + \tfrac{1}{2}] + c_{10}. \tag{47}$$

These values of α and β given by (47) and (46) satisfy (45), but a must be zero to satisfy (43).

Case (v). $a = a(\rho)$, $\varphi = 0$
Here also the values of α and β will be given by expressions (47) and (46) and as in case (iv) equations (43) and (25) yield $a = 0$.

Case (vi). $a = a(\rho, t)$, $\varphi = 0$
This case is similar to case (v) above.

3.2. *Static case with cosmological constant* Λ

Here we shall use the field equations with cosmological constant Λ. The appropriate equation is

$$R_\mu{}^\nu - \tfrac{1}{2}g_\mu{}^\nu R + \Lambda g_\mu{}^\nu = -\kappa T_\mu{}^\nu. \tag{48}$$

This yields the following field equations:

$$e^{2\beta-2\alpha}(\alpha_1/\rho - \beta_1^2) + \Lambda = \tfrac{1}{2}\kappa\lambda\, e^{2\beta-2\alpha}(\varphi^2 + a^2) \tag{49}$$

$$e^{2\beta-2\alpha}(\alpha_{11} + \beta_1^2) + \Lambda = \tfrac{1}{2}\kappa\lambda\, e^{2\beta-2\alpha}(\varphi^2 - a^2) \tag{50}$$

$$-e^{2\beta-2\alpha}[2(\beta_{11} + \beta_1/\rho) - (\alpha_{11} + \beta_1^2)] + \Lambda = \tfrac{1}{2}\kappa\lambda\, e^{2\beta-2\alpha}(\varphi^2 - a^2) \tag{51}$$

$$e^{2\beta-2\alpha}(\beta_1^2 - \alpha_1/\rho) + \Lambda = -\tfrac{1}{2}\kappa\lambda\, e^{2\beta-2\alpha}(\varphi^2 + a^2) \tag{52}$$

$$\kappa\lambda a\varphi\, e^{2\beta-2\alpha} = 0. \tag{53}$$

Adding (49) and (52) gives

$$\Lambda = 0. \tag{54}$$

3.3. *Non-static case with cosmological constant*

In this case where α, β, φ and a are functions of ρ and t only, (48) yields the following field equations:

$$e^{2\beta-2\alpha}(\alpha_1/\rho - \beta_1^2 - \beta_4^2) + \Lambda = \tfrac{1}{2}\kappa\lambda\, e^{2\beta-2\alpha}(\varphi^2 + a^2) \tag{55}$$

$$e^{2\beta-2\alpha}(\alpha_{11} - \alpha_{44} + \beta_1^2 - \beta_4^2) + \Lambda = \tfrac{1}{2}\kappa\lambda\, e^{2\beta-2\alpha}(\varphi^2 - a^2) \tag{56}$$

$$-e^{2\beta-2\alpha}[2(\beta_{11} - \beta_{44} + \beta_1/\rho) + (\alpha_{44} - \alpha_{11} + \beta_4^2 - \beta_1^2)] + \Lambda = \tfrac{1}{2}\kappa\lambda\, e^{2\beta-2\alpha}(\varphi^2 - a^2) \tag{57}$$

$$e^{2\beta-2\alpha}(\beta_1^2 + \beta_4^2 - \alpha_1/\rho) + \Lambda = -\tfrac{1}{2}\kappa\lambda\, e^{2\beta-2\alpha}(\varphi^2 + a^2) \tag{58}$$

$$-e^{2\beta-2\alpha}(2\beta_1\beta_4 - \alpha_4/\rho) = \kappa\lambda a\varphi\, e^{2\beta-2\alpha} \tag{59}$$

Adding (55) and (58), we obtain

$$\Lambda = 0. \tag{60}$$

In the perspective of renewed interest in the cosmological constant Λ in astrophysics, it is interesting to note that, under the assumptions made above, the metric does not allow the cosmological constant Λ in both static and non-static cases as shown in (54) and (60) above.

4. Geodesics of the field

We discuss here the equations governing the motion of a test particle in the field. Our assumption of negligible outflow of charge remains valid. Since we have taken $F_{\mu\nu} = 0$

to obtain the solutions of the field equations, the geodesic of a particle will be given by the equations

$$\frac{d^2x^i}{ds^2}+\left\{ {i \atop jk} \right\}\frac{dx^k}{ds}\frac{dx^j}{ds} = 0. \tag{61}$$

In our case, for $i = 2, 3, 4$ we get the equations of motion as

$$\frac{d}{ds}(\rho^2 e^{-2\beta}\dot{\phi}) = 0 \tag{62}$$

$$\frac{d}{ds}(e^{2\beta}\dot{z}) = 0 \tag{63}$$

$$\frac{d}{ds}(e^{2\alpha-2\beta}\dot{t}) = 0 \tag{64}$$

where dot denotes differentiation with respect to s. On integration, equations (62)–(64) give respectively

$$\rho^2 e^{-2\beta}\frac{d\phi}{ds} = A_1 \tag{65}$$

$$e^{2\beta}\frac{dz}{ds} = A_2 \tag{66}$$

$$e^{2\alpha-2\beta}\frac{dt}{ds} = A_3 \tag{67}$$

where A_1, A_2, A_3 are constants of integration.

In order to find out the first equation of the set for $i = 1$ and involving $\dot{\rho}$ we use the metric. This gives

$$\left(\frac{d\rho}{ds}\right)^2 = A_3^2 e^{-4\alpha+4\beta} - A_2^2 e^{-2\alpha} - \frac{A_1^2}{\rho^2} e^{-2\alpha+4\beta} - e^{-2\alpha+4\beta} \tag{68}$$

4.1. *Geodesic for the static solution*

In this section we give the expressions of the velocities in different directions and interpret the equations obtained. The values of α and β used in this case are given by (22) and (20).

Equations (65) and (67) give

$$\frac{d\phi}{dt} = \frac{A_1 e^{2\alpha}}{A_3\rho^2} = \frac{A_1}{A_3}\frac{e^{c_3}}{\rho^2} e^{2c_2 \ln \rho}.$$

Taking $c_2 = 1$, we obtain

$$\frac{d\phi}{dt} = \frac{A_1}{A_3} e^{c_3} = \text{constant}. \tag{69}$$

Similarly,

$$\frac{dz}{dt} = \frac{A_2}{A_3} e^{2c_3-4n}\rho^{2-4m}.$$

When $m = \frac{1}{2}$,

$$\frac{dz}{dt} = \frac{A_2}{A_3} e^{2c_3 - 4n} = \text{constant}. \tag{70}$$

Also,

$$\left(\frac{d\rho}{dt}\right)^2 = 1 - \frac{A_2^2}{A_3^2} e^{2c_3 - 4n} \rho^{2-4m} - \frac{A_1^2}{A_3^2} e^{2c_3} - \frac{1}{A_3^2} \rho^{2-2m} e^{2c_3 - 2n}$$
$$= 1 - k_1^2 - k_2^2 \rho^{2-4m} - k_3^2 \rho^{2-2m} \tag{71}$$

where k_1, k_2, and k_3 are new constants replacing $(A_1/A_3) e^{c_3}$, $(A_2/A_3) e^{c_3 - 2n}$ and $(1/A_3) e^{c_3 - n}$ respectively.

For $m = \frac{1}{2}$, (71) reduces to

$$\left(\frac{d\rho}{dt}\right)^2 = 1 - k_1^2 - k_2^2 - k_3^2 \rho. \tag{72}$$

This gives, on integration,

$$1 - k_1^2 - k_2^2 - k_3^2 \rho = \tfrac{1}{4}(k_3^2 t + k_4^2) \tag{73}$$

where k_4 is a constant. From (73) we have, on differentiation,

$$\frac{d^2\rho}{dt^2} = -\tfrac{1}{2} k_3^2. \tag{74}$$

Combining (69), (70) and (71) results in

$$V_\rho^2 = 1 - k_3^2 \rho - V_z - V_\phi^2 \tag{75}$$

where V_ρ, V_z and V_ϕ represent respectively the velocity along the radius, the velocity in the direction of z and the rate of describing the angle ϕ.

The radial velocity of the particle is not constant but depends on ρ, the distance from the axis of symmetry. Equation (74) shows that there exists a constant force acting on the test particle and tending towards the central axis. Thus the particle, free to move, will execute a helical path, taking the axis of symmetry as the axis of the helix. The helix will taper at the ends but the radial velocity will never be equal to zero unless $k_1^2 + k_2^2 = 1$.

This form of helical path may throw some light in explaining the formation of spiral nebulae seen in the universe.

4.2. Geodesic for the non-static solution

In this case the values of α and β are given by equations (41) and (36). We consider here the velocities of the particle in different directions when ϕ = constant. Equations (65)–(68) turn out to be

$$\frac{d\phi}{ds} = 0 \tag{76}$$

$$\frac{dz}{ds} = A_2 e^{-2\beta} \tag{77}$$

$$\frac{dt}{ds} = A_3 e^{-2\alpha + 2\beta} \tag{78}$$

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}s}\right)^2 = A_3^2\,\mathrm{e}^{-4\alpha+4\beta} - A_2^2\,\mathrm{e}^{-2\alpha} - \mathrm{e}^{-2\alpha+2\beta}. \tag{79}$$

Equations (78) and (79) yield

$$\begin{aligned}\left(\frac{\mathrm{d}\rho}{\mathrm{d}t}\right)^2 &= 1-\frac{A_2^2}{A_3^2}\mathrm{e}^{2\alpha-4\beta}-\frac{1}{A_3^2}\mathrm{e}^{2\alpha-2\beta}\\ &= 1-\frac{A_2^2}{A_3^2}\rho^2\exp[2(c_4t+c_5)(c_4t+c_5-2)]\\ &\quad\times\exp\{\tfrac{1}{2}k\lambda\varphi^2\rho^2+c_4^2\rho^2[(\ln\rho)^2-\ln\rho+\tfrac{1}{2}]+2c_8\}\\ &\quad-\frac{1}{A_3^2}\rho^2\exp[2(c_4t+c_5)(c_4t+c_5-1)]\\ &\quad\times\exp\{\tfrac{1}{2}k\lambda\varphi^2\rho^2+c_4^2\rho^2[(\ln\rho)^2-\ln\rho+\tfrac{1}{2}]+2c_8\}\end{aligned} \tag{80}$$

and

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{A^2}{A_3}\rho^{2(c_4t+c_5)(c_4t+c_5-2)}\exp\{\tfrac{1}{2}k\lambda\varphi^2\rho^2+c_4^2\rho^2[(\ln\rho)^2-\ln\rho+\tfrac{1}{2}]+2c_8\}. \tag{81}$$

Equations (80) and (81) give the velocity of the particle along the radius and along the direction of the z axis respectively.

5. Regularity of the solutions

Here we study the regularity of the solutions and we see that the solutions are not everywhere regular. Following Bonnor (1957) we define a non-singular field as one in which every point (including points at spatial and temporal infinity) is non-singular. A point P is non-singular if at this point natural coordinates can be introduced by means of a coordinate transformation for which a set of sufficient conditions is:

(i) g, the determinant of the metric tensor g_{ij}, is nonzero;
(ii) g_{ij} and their first derivatives are finite and continuous at P; (82)
(iii) the second derivatives of g_{ij} are finite and continuous at P.

To avoid the coordinate singularity we transform the metric (8) to the pseudo-Cartesian coordinates by setting

$$x=\rho\cos\phi,\qquad y=\rho\sin\phi,\qquad z=z,\qquad t=t.$$

By this transformation the metric (8) becomes

$$\begin{aligned}\mathrm{d}s^2 &= -\mathrm{e}^{2\beta}\,\mathrm{d}z^2-\rho^{-2}[(x^2\,\mathrm{e}^{2\alpha-2\beta}+y^2\,\mathrm{e}^{-2\beta})\,\mathrm{d}x^2+(y^2\,\mathrm{e}^{2\alpha-2\beta}+x^2\,\mathrm{e}^{-2\beta})\,\mathrm{d}y^2]\\ &\quad+2xy(\mathrm{e}^{-2\beta}-\mathrm{e}^{2\alpha-2\beta})\,\mathrm{d}x\,\mathrm{d}y+\mathrm{e}^{2\alpha-2\beta}\,\mathrm{d}t^2.\end{aligned} \tag{83}$$

It can now be easily checked that the solutions given by (19), (22) and (36), (41) do not satisfy the condition (82) of regularity and as such the solutions are not everywhere regular. In fact, all of them are singular along the z axis and also at infinity.

We shall now discuss whether the solutions represent the field of a line-mass. For solutions (19) and (22),

$$g_{44} = -g_{11} = \mathrm{e}^{2\alpha-2\beta} = \text{constant}\times\rho^{2(c_2-m)}. \tag{84}$$

For solutions (36) and (41) and for t = constant hypersurface,

$$g_{44} = -g_{11} = e^{2\alpha - 2\beta}$$

$$= \text{constant} \times \rho^h \exp\{c_4^2\rho^2[(\ln\rho)^2 - \ln\rho + \tfrac{1}{2}]\}\left(1 + \frac{\kappa\lambda\varphi^2\rho^2}{2.1!} + \ldots + O(\rho^2)\right) \quad (85)$$

where h is a constant. In both cases expressions (84) and (85) do not represent the field of a line-mass.

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